Fixed Effects Binary Choice Models with Three or More Periods
Online Appendix

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We detail here the last step of the proof of Proposition 2.6 of "Point Identification of Panel Binary Models Without Logit Errors". Specifically, let us show that if

$$\begin{aligned}
&(\nu_{11}+\nu_{12})e^{a_1 v}+\alpha_1(\nu_{11}e^{\lambda_{01}\gamma_2}+\nu_{12}e^{\lambda_{01}\gamma_1})e^{(a_1+b_1)v}+\alpha_2(\nu_{11}e^{\lambda_{02}\gamma_2}+\nu_{12}e^{\lambda_{02}\gamma_1})e^{(a_1+b_2)v}\\
&+(\nu_{21}+\nu_{22})e^{a_2 v}+\alpha_1(\nu_{21}e^{\lambda_{01}\gamma_2}+\nu_{22}e^{\lambda_{01}\gamma_1})e^{(a_2+b_1)v}+\alpha_2(\nu_{21}e^{\lambda_{02}\gamma_2}+\nu_{22}e^{\lambda_{02}\gamma_1})e^{(a_2+b_2)v}\\
&+(\nu_{31}+\nu_{32})e^{b_1 v}+\alpha_1(\nu_{31}e^{\lambda_{01}\gamma_2}+\nu_{32}e^{\lambda_{01}\gamma_1})e^{2b_1 v}+\alpha_2(\nu_{31}e^{\lambda_{02}\gamma_2}+\nu_{32}e^{\lambda_{02}\gamma_1})e^{(b_1+b_2)v}\\
&+(\nu_{41}+\nu_{42})e^{b_2}+\alpha_1(\nu_{41}e^{\lambda_{01}\gamma_2}+\nu_{42}e^{\lambda_{01}\gamma_1})e^{(b_1+b_2)v}+\alpha_2(\nu_{41}e^{\lambda_{02}\gamma_2}+\nu_{42}e^{\lambda_{02}\gamma_1})e^{2b_2 v}=0,
\end{aligned} \tag{1}$$

for all $v \in \mathbb{R}$, then $\boldsymbol{\nu} := (\nu_{11}, \nu_{12}, \nu_{21}, \nu_{22}, \nu_{31}, \nu_{32}, \nu_{41}, \nu_{42})' = \mathbf{0}$. The proof consists in using Lemma B.1 (see the main paper) to obtain a sufficient number of restrictions on the coefficients of the exponential polynomial appearing in (1) given all possible values of the a_ℓ and b_ℓ. In the most favorable case where

$$|\{a_1, a_2, b_1, b_2, a_1+b_1, a_1+b_2, a_2+b_1, a_2+b_2, 2b_1, b_1+b_2, 2b_2\}| = 11, \tag{2}$$

Lemma B.1 can readily be applied and, once combined with $\gamma_1 \neq \gamma_2$, yields the result. We show that it actually holds in any permissible case (i.e. under any value for $(\beta_k, \beta_{0k}, \lambda_0)$ that is consistent with the model assumptions). To do so, we consider

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all possible orderings of $\{a_1, a_2, b_1, b_2\}$. The latter are only partly determined by the sign of β_k, β_{0k}. For instance, if $\beta_{0k} > 0$ and $\beta_k < 0$ then $b_1 > b_1 + a_2$ but the ordering of $\{b_1, b_2 + a_1\}$ remains unknown. Hence, among each possible ordering, we further adress each possible case of equality between the term appearing in (2). Hereafter, the symbol "$\stackrel{?}{=}$" represents those cases of potential equality that cannot be ruled out by simply restricting the signs of β_k, β_{0k}. In what follows, small or capital letters and numeric or literal numbers are used to label these equalities. Notice that $\Lambda_\tau \subset \{(1, \lambda_2) : \lambda_2 > 4\}$ implies

$$a_2/a_1 = b_2/b_1 = \lambda_{02} \notin \left\{4/3, 3/2, 2, \frac{3+\sqrt{5}}{2}, 3, 4\right\}. \tag{3}$$

Case 1 : $\beta_{0k} > 0$ and $\beta_k < 0$. From $\lambda_{02} > \lambda_{01}$, we necessarily have $a_2 < a_1 < 0 < b_1 < b_2$. Together with (3), we obtain the following order relations.

$$\begin{array}{ll} b_1 & < b_1 + b_2 \\ & < 2b_2 \\ & < 2b_1 \\ & > b_1 + a_1 \\ & > b_1 + a_2 \\ & \stackrel{?}{=} b_2 + a_1 \quad (i.) \\ & \stackrel{?}{=} b_2 + a_2 \quad (ii.) \\ & < b_2 \\ & > a_1 \\ & > a_2 \end{array}, \qquad \begin{array}{ll} 2b_1 & < b_1 + b_2 \\ & < 2b_2 \\ & > b_1 \\ & > b_1 + a_1 \\ & > b_1 + a_2 \\ & \stackrel{?}{=} b_2 + a_1 \quad (iii.) \\ & \stackrel{?}{=} b_2 + a_2 \quad (iv.) \\ & \neq b_2 \\ & > a_1 \\ & > a_2 \end{array}, \qquad \begin{array}{ll} b_1 + b_2 & > \lambda_1 \\ & > b_2 \\ & > a_1 \\ & > a_2 \\ & < 2b_2 \\ & > 2b_1 \\ & > b_1 + a_1 \\ & > b_1 + a_2 \\ & > b_2 + a_1 \\ & > b_2 + a_2 \end{array}, \tag{4}$$

and

$$\begin{aligned} b_2 &< b_1+b_2 \\ &< 2b_2 \\ &\neq 2b_1 \\ &> b_1+a_1 \\ &> b_1+a_2 \\ &> b_2+a_1 \\ &> b_2+a_2 \\ &> b_1 \\ &> a_1 \\ &> a_2 \end{aligned}, \qquad \begin{aligned} 2b_2 &> b_1+b_2 \\ &> 2b_1 \\ &> b_1 \\ &> b_1+a_1 \\ &> b_1+a_2 \\ &> b_2+a_1 \\ &> b_2+a_2 \\ &> b_2 \\ &> a_1 \\ &> a_2 \end{aligned}, \qquad \begin{aligned} b_1+b_2 &> b_1 \\ &> b_2 \\ &> a_1 \\ &> a_2 \\ &< 2b_2 \\ &> 2b_1 \\ &> b_1+a_1 \\ &> b_1+a_2 \\ &> b_2+a_1 \\ &> b_2+a_2 \end{aligned}, \tag{5}$$

and

$$\begin{aligned} a_1 &< b_1 \\ &< b_2 \\ &< b_1+b_2 \\ &< 2b_2 \\ &< 2b_1 \\ &< b_1+a_1 \\ &\stackrel{?}{=} b_1+a_2 \quad (v.) \\ &< b_2+a_1 \\ &\stackrel{?}{=} b_2+a_2 \quad (vi.) \\ &> a_2 \end{aligned}, \qquad \begin{aligned} b_1+a_1 &< b_1+b_2 \\ &< 2b_2 \\ &< 2b_1 \\ &< b_1 \\ &> b_1+a_2 \\ &< b_2+a_1 \\ &\stackrel{?}{=} b_2+a_2 \quad (vii.) \\ &< b_2 \\ &> a_1 \\ &> a_2 \end{aligned}, \qquad \begin{aligned} b_2+a_1 &< b_1+b_2 \\ &\stackrel{?}{=} b_1 \quad (i.) \\ &< b_2 \\ &> a_1 \\ &> a_2 \\ &< 2b_2 \\ &\stackrel{?}{=} 2b_1 \quad (iii.) \\ &> b_1+a_1 \\ &> b_1+a_2 \\ &> b_2+a_2 \end{aligned}, \tag{6}$$

and

$$
\begin{aligned}
a_2 &< b_1 \\ &< b_2 \\ &< b_1+b_2 \\ &< 2b_2 \\ &< 2b_1 \\ &< b_1+a_1 \\ &< b_1+a_2 \\ &< b_2+a_1 \\ &< b_2+a_2 \\ &< a_1
\end{aligned}
\quad , \quad
\begin{aligned}
b_1+a_2 &< b_1+b_2 \\ &< 2b_2 \\ &< b_1 \\ &< b_1+a_1 \\ &< b_2+a_1 \\ &< b_2+a_2 \\ &< b_2 \\ &< 2b_1 \\ &\stackrel{?}{=} a_1 \quad (v.) \\ &> a_2
\end{aligned}
\quad , \quad
\begin{aligned}
b_2+a_2 &< b_1+b_2 \\ &\stackrel{?}{=} b_1 \quad (ii.) \\ &< b_2 \\ &\stackrel{?}{=} a_1 \quad (vi.) \\ &> a_2 \\ &< 2b_2 \\ &\stackrel{?}{=} 2b_1 \quad (iv.) \\ &\stackrel{?}{=} b_1+a_1 \quad (vii.) \\ &> b_1+a_2 \\ &< b_2+a_1
\end{aligned}
\quad . \tag{7}
$$

Lemma B.1 combined with equation (1) and inequalities (5) implies that the coefficients in front of $e^{b_2 v}$ and $e^{2b_2 v}$ in equation (1) are zero. It follows from $\gamma_1 \neq \gamma_2$ that $\nu_{41} = \nu_{42} = 0$. There remain seven subcases to deal with.

1. Suppose $i.$ holds. Then, none of the following equalities hold : $ii.$ (else one would have $a_1 = a_2 \implies \lambda_2 = 1$ and obtain a contradiction), $iii.$ (else one would have $b_1 = 2b_1 \implies \beta_{0k} = 0$ and obtain a contradiction), $iv.$ (else one would have $a_2/a_1 = 2$ and obtain a contradiction with (3)). Also, it can be verified by using exactly the same kind of arguments that at most one equation among $v.$, $vi.$ and $vii.$ holds. Suppose that $i.$ and $vi.$ hold, then Lemma B.1 combined with equation (1) and inequalities (4) implies that the coefficients in front of $e^{2b_1 v}$ and $e^{(b_1+b_2)v}$ in equation (1) are zero. It follows from $\gamma_1 \neq \gamma_2$ that $\nu_{31} = \nu_{32} = 0$. Hence, equation (1) is free from exponential monomials whose exponents have only terms in b_1, b_2 so that we can ignore equality $i.$ in inequalities (6) to obtain, again by Lemma B.1 and equation (1), that the coefficients in front of $e^{(a_1+b_1)v}$ and $e^{(a_1+b_2)v}$ are zero. It follows from $\gamma_1 \neq \gamma_2$ that $\nu_{11} = \nu_{12} = 0$. The same reasoning now trivially implies that the coefficients in front of $e^{a_2 v}$ and $e^{(a_2+b_1)v}$ are zero. It follows from $\gamma_1 \neq \gamma_2$ that $\nu_{21} = \nu_{22} = 0$, i.e. $\boldsymbol{\nu} = 0$. The proof for the other pairs of equalities of this subcase ($i.$ and $v.$, or $i.$ and $vii.$) is almost identical and is thus omitted. Similarly, the rest of the proof for the subsequent other subcases and other cases hinges on the same arguments and

we omit repeating the entire reasonning developped here.

2. Suppose *ii.* holds. Then, none of *i.*, *iv.*, *vi.* and *vii.* hold. If *v.* does not hold then the result follows. If *v.* holds, then equation *iii.* also holds as it is equivalent to *v.* when *ii.* holds and we have

$$\begin{cases} a_2 &= b_1 - b_2, \\ a_1 &= 2b_1 - b_2. \end{cases} \tag{8}$$

The unique pairwise different exponents of the exponential polynomial in (1) are: $b_1 + b_2$, $2b_1 (= b_2 + a_1)$, $b_1 (= b_2 + a_2)$, $a_1 (= b_1 + a_2)$, $b_1 + a_1$ and a_2. This puts 6 restrictions over the six unknown parameters $\boldsymbol{\nu} \backslash \{\nu_{41}, \nu_{42}\}$:

$$\begin{pmatrix} 0 & 0 & 0 & 0 & \alpha_2 e^{\lambda_2\gamma_2} & \alpha_2 e^{\lambda_2\gamma_1} \\ \alpha_2 e^{\lambda_2\gamma_2} & \alpha_2 e^{\lambda_2\gamma_1} & 0 & 0 & \alpha_1 e^{\lambda_1\gamma_2} & \alpha_1 e^{\lambda_1\gamma_1} \\ 0 & 0 & \alpha_2 e^{\lambda_2\gamma_2} & \alpha_2 e^{\lambda_2\gamma_1} & 1 & 1 \\ 1 & 1 & \alpha_1 e^{\lambda_1\gamma_2} & \alpha_1 e^{\lambda_1\gamma_1} & 0 & 0 \\ \alpha_1 e^{\lambda_1\gamma_2} & \alpha_1 e^{\lambda_1\gamma_1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} \nu_{11} \\ \nu_{12} \\ \nu_{21} \\ \nu_{22} \\ \nu_{31} \\ \nu_{32} \end{pmatrix} = \mathbf{0}.$$

A tedious computation shows that the determinant of the above matrix is zero meaning that this system of linear equations has non trivial solutions. However, notice that (8) cannot hold since $\beta_k \beta_{0k} < 0$ implies

$$(8) \iff \begin{cases} \lambda_{02}\beta_k &= (1 - \lambda_{02})\beta_{0k} \\ \beta_k &= (2 - \lambda_{02})\beta_{0k} \end{cases} \implies \begin{cases} -\lambda_{02}^2 + 3\lambda_{02} - 1 &= 0 \\ 1 < \lambda_{02} < 2 \end{cases},$$

which has an empty solution set.

3. Suppose *iii.* holds, i.e. $a_1 = 2b_1 - b_2$. Then, none of *i.*, *iv.* hold and at most one among *ii.*, *v.*, *vi.* and *vii.* holds. The result follows.

4. Suppose *iv.* holds, i.e. $a_2 = 2b_1 - b_2$. Then, none of *i.*, *ii.*, *iii.*, *v.*, *vi.* and *vii.* hold and the result follows.

5. Suppose *v.* holds, i.e. $a_2 = a_1 - b_1$. Then, none of *iv.*, *vi.* and *vii.* hold. At most one among *iii.* = *ii.* and *i.* holds. If *ii.* does not hold, the result follows. If *ii.* holds, then,

$$\begin{cases} a_2 &= b_1 - b_2, \\ a_1 &= 2b_1 - b_2, \end{cases}$$

and we are back to subcase 2.

6. Suppose *vi*. holds, i.e. $a_2 = a_1 - b_2$. Then, none of *ii*., *iv*., *v*. and *vii*. hold. At most one among *i*. and *iii*. holds. The result follows.

7. Suppose *vii*. holds, i.e. $a_2 = a_1 + b_1 - b_2$. Then, none of *ii*., *iv*., *v*. and *vi*. hold. At most one among *i*. and *iii*. holds. The result follows.

Case 2 : $\beta_{0k} > 0$ and $\beta_k > 0$. Notice that for all $(i, j) \in \{(1, 2), (2, 1)\}$,

$$(a_i < b_i \wedge a_j > b_j) \implies (\beta_k < \beta_{0k} \wedge \beta_k > \beta_{0k}),$$

a contradiction. Hence, there are only four permissible orderings of $\{a_1, a_2, b_1, b_2\}$.

- First, suppose $0 < b_1 < a_1 < b_2 < a_2$. Then, we have

$$\begin{aligned} b_1 &< b_1 + b_2 \\ &< 2b_2 \\ &< 2b_1 \\ &< b_1 + a_1 \\ &< b_1 + a_2 \\ &< b_2 + a_1 \\ &< b_2 + a_2 \\ &< b_2 \\ &< a_1 \\ &< a_2 \end{aligned}, \qquad \begin{aligned} 2b_1 &< b_1 + b_2 \\ &< 2b_2 \\ &> b_1 \\ &< b_1 + a_1 \\ &< b_1 + a_2 \\ &< b_2 + a_1 \\ &< b_2 + a_2 \\ &\neq b_2 \\ &\stackrel{?}{=} a_1 \quad (I.) \\ &\stackrel{?}{=} a_2 \quad (II.) \end{aligned}, \qquad \begin{aligned} b_1 + b_2 &> b_1 \\ &> b_2 \\ &> a_1 \\ &\stackrel{?}{=} a_2 \quad (III.) \\ &< 2b_2 \\ &> 2b_1 \\ &> b_1 + a_1 \\ &< b_1 + a_2 \\ &< b_2 + a_1 \\ &< b_2 + a_2 \end{aligned}, \quad (9)$$

and

$$
\begin{aligned}
b_2 \quad &< b_1+b_2 \\
&< 2b_2 \\
&\neq 2b_1 \\
&\stackrel{?}{=} b_1+a_1 \quad (IV.) \\
&< b_1+a_2 \\
&< b_2+a_1 \\
&< b_2+a_2 \\
&> b_1 \\
&> a_1 \\
&< a_2
\end{aligned}
\quad , \quad
\begin{aligned}
2b_2 \quad &> b_1+b_2 \\
&> 2b_1 \\
&> b_1 \\
&> b_1+a_1 \\
&\stackrel{?}{=} b_1+a_2 \quad (V.) \\
&> b_2+a_1 \\
&< b_2+a_2 \\
&> b_2 \\
&> a_1 \\
&\stackrel{?}{=} a_2 \quad (VI.)
\end{aligned}
\quad , \quad
\begin{aligned}
b_1+b_2 \quad &> b_1 \\
&> b_2 \\
&> a_1 \\
&\stackrel{?}{=} a_2 \quad (III.) \\
&< 2b_2 \\
&> 2b_1 \\
&> b_1+a_1 \\
&< b_1+a_2 \\
&< b_2+a_1 \\
&< b_2+a_2
\end{aligned}
\quad , \tag{10}
$$

and

$$
\begin{aligned}
a_1 \quad &> b_1 \\
&< b_2 \\
&< b_1+b_2 \\
&< 2b_2 \\
&\stackrel{?}{=} 2b_1 \quad (I.) \\
&< b_1+a_1 \\
&< b_1+a_2 \\
&< b_2+a_1 \\
&< b_2+a_2 \\
&< a_2
\end{aligned}
\quad , \quad
\begin{aligned}
b_1+a_1 \quad &< b_1+b_2 \\
&< 2b_2 \\
&> 2b_1 \\
&> b_1 \\
&< b_1+a_2 \\
&< b_2+a_1 \\
&< b_2+a_2 \\
&\stackrel{?}{=} b_2 \quad (IV.) \\
&> a_1 \\
&\stackrel{?}{=} a_2 \quad (VII.)
\end{aligned}
\quad , \quad
\begin{aligned}
b_2+a_1 \quad &> b_1+b_2 \\
&> b_1 \\
&> b_2 \\
&> a_1 \\
&\stackrel{?}{=} a_2 \quad (VIII.) \\
&< 2b_2 \\
&> 2b_1 \\
&> b_1+a_1 \\
&\stackrel{?}{=} b_1+a_2 \quad (IX.) \\
&< b_2+a_2
\end{aligned}
\quad , \tag{11}
$$

and

$$
\begin{aligned}
a_2 &> b_1 \\
&> b_2 \\
&\stackrel{?}{=} b_1 + b_2 \quad (III) \\
&\stackrel{?}{=} 2b_2 \quad (VI.) \\
&\stackrel{?}{=} 2b_1 \quad (II.) \\
&\stackrel{?}{=} b_1 + a_1 \quad (VII.) \\
&< b_1 + a_2 \\
&\stackrel{?}{=} b_2 + a_1 \quad (VIII.) \\
&< b_2 + a_2 \\
&> a_1
\end{aligned}
\quad , \quad
\begin{aligned}
b_1 + a_2 &> b_1 + b_2 \\
&\stackrel{?}{=} 2b_2 \quad (V.) \\
&> b_1 \\
&> b_1 + a_1 \\
&\stackrel{?}{=} b_2 + a_1 \quad (IX.) \\
&< b_2 + a_2 \\
&> b_2 \\
&> 2b_1 \\
&> a_1 \\
&> a_2
\end{aligned}
\quad , \quad
\begin{aligned}
b_2 + a_2 &> b_1 + b_2 \\
&> b_1 \\
&> b_2 \\
&> a_1 \\
&> a_2 \\
&> 2b_2 \\
&> 2b_1 \\
&> b_1 + a_1 \\
&> b_1 + a_2 \\
&> b_2 + a_1
\end{aligned}
\quad . \tag{12}
$$

There are nine subcases left.

1. $I.$ holds, i.e. $a_1 = 2b_1$. Then, $II.$, $IV.$ or $VII.$ cannot hold. At most one among $IX. = III.$, $V.$, $VI.$ and $VIII.$ holds. The only problematic case is when $IX. = III.$ holds. In this case, we deduce from equation (10) and Lemma B.1 that $\nu_{41} = \nu_{42} = 0$. Next, we have

$$
\begin{cases}
a_1 &= 2b_1, \\
a_2 &= b_1 + b_2.
\end{cases} \tag{13}
$$

The unique pairwise different exponents of the exponential polynomial in (1) are: b_1, $2b_1 (= a_1)$, $b_1 + b_2 (= a_2)$, $b_1 + a_1$, $b_2 + a_1 (= b_1 + a_2)$ and $b_2 + a_2$. This puts 6 restrictions over the six unknown parameter $\boldsymbol{\nu} \backslash \{\nu_{41}, \nu_{42}\}$:

$$
\begin{pmatrix}
0 & 0 & 0 & 0 & 1 & 1 \\
1 & 1 & 0 & 0 & \alpha_1 e^{\lambda_1 \gamma_2} & \alpha_1 e^{\lambda_1 \gamma_1} \\
0 & 0 & 1 & 1 & \alpha_2 e^{\lambda_2 \gamma_2} & \alpha_2 e^{\lambda_2 \gamma_1} \\
\alpha_1 e^{\lambda_1 \gamma_2} & \alpha_1 e^{\lambda_1 \gamma_1} & 0 & 0 & 0 & 0 \\
\alpha_2 e^{\lambda_2 \gamma_2} & \alpha_2 e^{\lambda_2 \gamma_1} & \alpha_1 e^{\lambda_1 \gamma_2} & \alpha_1 e^{\lambda_1 \gamma_1} & 0 & 0 \\
0 & 0 & \alpha_2 e^{\lambda_2 \gamma_2} & \alpha_2 e^{\lambda_2 \gamma_1} & 0 & 0
\end{pmatrix}
\cdot
\begin{pmatrix}
\nu_{11} \\ \nu_{12} \\ \nu_{21} \\ \nu_{22} \\ \nu_{31} \\ \nu_{32}
\end{pmatrix}
= \mathbf{0}.
$$

A tedious computation shows that the determinant of the above matrix is zero meaning that this system of linear equations has non trivial solutions.

However, notice that (13) cannot hold since

$$(13) \iff \begin{cases} \beta_k &= 2\beta_{0k} \\ \lambda_{02}\beta_k &= (1+\lambda_{02})\beta_{0k} \end{cases} \implies \lambda_{02} = 1,$$

and $(1,1) \notin \Lambda_\tau$.

2. Suppose $II.$ holds, i.e. $a_2 = 2b_1$. Then, none of $I.$, $III.$, $V.$, $VI.$, $VII.$ hold. At most one among $IV.$, $VIII.$, $IX.$ holds. The result follows.

3. Suppose $III.$ holds, i.e. $a_2 = b_2 - b_1$. Then, none of $II.$, $V.$ and $VI.$ hold. At most one among $IV.$, $VII.$, $VIII.$, $IX.$ holds. The result follows.

4. Suppose $IV.$ holds, i.e $a_1 = b_2 - b_1$. Then, none of $I.$, $VII.$ hold. At most one among $III.$, $V. = VIII.$, $VI.$, $IX.$ holds. By combining equation (9) and Lemma B.1, we obtain $\nu_{31} = \nu_{32} = 0$. The only problematic case is when $V.$ holds. Then,

$$\begin{cases} a_1 &= b_2 - b_1, \\ a_2 &= 2b_2 - b_1. \end{cases} \tag{14}$$

The unique pairwise different exponents of the exponential polynomial in (1) are: $b_2(= b_1 + a_1)$, $2b_2(= b_1 + a_2)$, $b_1 + b_2$, a_1, $b_2 + a_1(= a_2)$, and $b_2 + a_2$. This puts 6 restrictions over the six unknown parameter $\boldsymbol{\nu}\backslash\{\nu_{31}, \nu_{32}\}$:

$$\begin{pmatrix} \alpha_1 e^{\lambda_1\gamma_2} & \alpha_1 e^{\lambda_1\gamma_1} & 0 & 0 & 1 & 1 \\ 0 & 0 & \alpha_1 e^{\lambda_1\gamma_2} & \alpha_1 e^{\lambda_1\gamma_1} & \alpha_2 e^{\lambda_2\gamma_2} & \alpha_2 e^{\lambda_2\gamma_1} \\ 0 & 0 & 0 & 0 & \alpha_1 e^{\lambda_1\gamma_2} & \alpha_1 e^{\lambda_1\gamma_1} \\ 1 & 1 & 0 & 0 & 0 & 0 \\ \alpha_2 e^{\lambda_2\gamma_2} & \alpha_2 e^{\lambda_2\gamma_1} & 1 & 1 & 0 & 0 \\ 0 & 0 & \alpha_2 e^{\lambda_2\gamma_2} & \alpha_2 e^{\lambda_2\gamma_1} & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} \nu_{11} \\ \nu_{12} \\ \nu_{21} \\ \nu_{22} \\ \nu_{41} \\ \nu_{42} \end{pmatrix} = \mathbf{0}.$$

A tedious computation shows that the determinant of the above matrix is zero meaning that this system of linear equations has non trivial solutions.

However, notice that (14) cannot hold since

$$
\begin{aligned}
(14) \iff & \begin{cases} \beta_k & = (\lambda_{02}-1)\beta_{0k} \\ \lambda_{02}\beta_k & = (2\lambda_{02}-1)\beta_{0k} \end{cases} \\
\implies & \begin{cases} \lambda_{02}^2 - 3\lambda_{02} + 1 = 0 & \\ 2 < \lambda_{02} & \text{(because } \beta_{0k} < \beta_k\text{)} \\ -\lambda_{02}^2 + 2\lambda_{02} - 1 < 0 & \text{(because } \beta_k < \lambda_{02}\beta_{0k}\text{)} \end{cases} \\
\implies & \lambda_{02} = (3+\sqrt{5})/2,
\end{aligned}
$$

and $\left(1, \frac{3+\sqrt{5}}{2}\right) \notin \Lambda_\tau$.

5. Suppose $V.$ holds, i.e. $a_2 = 2b_2 - b_1$. Then, none of $II.$, $III.$, $VI.$ and $IX.$ hold. At most one among $I.$, $IV.$, $VII.$, $VIII.$ holds or ($IV.$ and $VIII. = V.$) hold. Suppose $IV.$ and $VIII. = V.$ hold. Then, $\nu_{31} = \nu_{32} = 0$ from equation (9) and Lemma B.1. We have

$$
\begin{cases} a_1 & = b_2 - b_1, \\ a_2 & = 2b_2 - b_1. \end{cases} \tag{15}
$$

and we are back to subcase 4.

6. Suppose $VI.$ holds, i.e $a_2 = 2b_2$. Then, none of $II.$, $III.$, $VI.$ hold. At most one among $I.$, $VII.$, $VIII.$ and $IX.$ holds. The result follows.

7. Suppose $VII.$ holds, i.e. $a_2 = b_1 + a_1$. Then, none of I, II, $IV.$, $VIII.$ and $IX.$ hold. At most one among $III.$, $V.$ and $VI.$ holds. The result follows.

8. Suppose $VIII.$ holds, i.e. $a_2 = b_2 + a_1$. Then, none of $III.$, $VI.$, $VII.$ and $IX.$ hold. At most one among $I.$, $II.$, $IV.$, $V.$ holds or $IV. = V.$ holds. Consider the second case. Then, $\nu_{31} = \nu_{32} = 0$ from equation (9) and Lemma B.1. We have

$$
\begin{cases} a_1 & = b_2 - b_1, \\ a_2 & = 2b_2 - b_1. \end{cases} \tag{16}
$$

and we are back to subcase 4.

9. Suppose $IX.$ holds, i.e. $a_2 = b_2 - b_1 + a_1$. Then, none of $III.$, $V.$, $VII.$ and $VIII.$ hold. At most one among $I.$, $II.$, $IV.$ and $VI.$ holds. The result follows.

- Second, suppose $0 < b_1 < b_2 < a_1 < a_2$. Then, we have

$$\begin{aligned} b_1 &< b_1 + b_2 \\ &< 2b_2 \\ &< 2b_1 \\ &< b_1 + a_1 \\ &< b_1 + a_2 \\ &< b_2 + a_1 \\ &< b_2 + a_2 \\ &< b_2 \\ &< a_1 \\ &< a_2 \end{aligned}, \qquad \begin{aligned} 2b_1 &< b_1 + b_2 \\ &< 2b_2 \\ &> b_1 \\ &< b_1 + a_1 \\ &< b_1 + a_2 \\ &< b_2 + a_1 \\ &< b_2 + a_2 \\ &\neq b_2 \\ &\stackrel{?}{=} a_1 \quad (\widetilde{I.}) \\ &\stackrel{?}{=} a_2 \quad (\widetilde{II.}) \end{aligned}, \qquad \begin{aligned} b_1 + b_2 &> b_1 \\ &> b_2 \\ &\stackrel{?}{=} a_1 \quad (\widetilde{III.}) \\ &\stackrel{?}{=} a_2 \quad (\widetilde{IV.}) \\ &< 2b_2 \\ &> 2b_1 \\ &< b_1 + a_1 \\ &< b_1 + a_2 \\ &< b_2 + a_1 \\ &< b_2 + a_2 \end{aligned}, \tag{17}$$

and

$$\begin{aligned} b_2 &< b_1 + b_2 \\ &< 2b_2 \\ &\neq 2b_1 \\ &< b_1 + a_1 \\ &< b_1 + a_2 \\ &< b_2 + a_1 \\ &< b_2 + a_2 \\ &> b_1 \\ &< a_1 \\ &< a_2 \end{aligned}, \qquad \begin{aligned} 2b_2 &> b_1 + b_2 \\ &> 2b_1 \\ &> b_1 \\ &\stackrel{?}{=} b_1 + a_1 \quad (\widetilde{V.}) \\ &\stackrel{?}{=} b_1 + a_2 \quad (\widetilde{VI.}) \\ &\neq b_2 + a_1 \\ &< b_2 + a_2 \\ &\neq b_2 \\ &\stackrel{?}{=} a_1 \quad (\widetilde{VII.}) \\ &\stackrel{?}{=} a_2 \quad (\widetilde{VIII.}) \end{aligned}, \qquad \begin{aligned} b_1 + b_2 &> b_1 \\ &> b_2 \\ &\stackrel{?}{=} a_1 \quad (\widetilde{III.}) \\ &\stackrel{?}{=} a_2 \quad (\widetilde{IV.}) \\ &< 2b_2 \\ &> 2b_1 \\ &< b_1 + a_1 \\ &< b_1 + a_2 \\ &< b_2 + a_1 \\ &< b_2 + a_2 \end{aligned}, \tag{18}$$

and

$$\begin{aligned} a_1 &> b_1 \\ &> b_2 \\ &\overset{?}{=} b_1 + b_2 \quad (\widetilde{III}.) \\ &\overset{?}{=} 2b_2 \quad (\widetilde{VII}.) \\ &\overset{?}{=} 2b_1 \quad (\widetilde{I}.) \\ &< b_1 + a_1 \\ &< b_1 + a_2 \\ &< b_2 + a_1 \\ &< b_2 + a_2 \\ &< a_2 \end{aligned}, \qquad \begin{aligned} b_1 + a_1 &\neq b_1 + b_2 \\ &\overset{?}{=} 2b_2 \quad (\widetilde{V}.) \\ &> 2b_1 \\ &> b_1 \\ &< b_1 + a_2 \\ &< b_2 + a_1 \\ &< b_2 + a_2 \\ &> b_2 \\ &> a_1 \\ &\overset{?}{=} a_2 \quad (\widetilde{IX}.) \end{aligned}, \qquad \begin{aligned} b_2 + a_1 &> b_1 + b_2 \\ &> b_1 \\ &> b_2 \\ &> a_1 \\ &\overset{?}{=} a_2 \quad (\widetilde{X}) \\ &> 2b_2 \\ &> 2b_1 \\ &> b_1 + a_1 \\ &\overset{?}{=} b_1 + a_2 \quad (\widetilde{XI}.) \\ &< b_2 + a_2 \end{aligned}, \tag{19}$$

and

$$\begin{aligned} a_2 &> b_1 \\ &> b_2 \\ &\overset{?}{=} b_1 + b_2 \quad (\widetilde{IV}.) \\ &\overset{?}{=} 2b_2 \quad (\widetilde{VIII}.) \\ &\overset{?}{=} 2b_1 \quad (\widetilde{II}.) \\ &\overset{?}{=} b_1 + a_1 \quad (\widetilde{IX}.) \\ &< b_1 + a_2 \\ &\overset{?}{=} b_2 + a_1 \quad (\widetilde{X}.) \\ &< b_2 + a_2 \\ &> a_1 \end{aligned}, \qquad \begin{aligned} b_1 + a_2 &\neq b_1 + b_2 \\ &\overset{?}{=} 2b_2 \quad (\widetilde{VI}.) \\ &> b_1 \\ &> b_1 + a_1 \\ &\overset{?}{=} b_2 + a_1 \quad (\widetilde{XI}.) \\ &< b_2 + a_2 \\ &> b_2 \\ &\neq 2b_1 \\ &> a_1 \\ &> a_2 \end{aligned}, \qquad \begin{aligned} b_2 + a_2 &> b_1 + b_2 \\ &> b_1 \\ &> b_2 \\ &> a_1 \\ &> a_2 \\ &> 2b_2 \\ &> 2b_1 \\ &> b_1 + a_1 \\ &> b_1 + a_2 \\ &> b_2 + a_1 \end{aligned}. \tag{20}$$

There are eleven subcases left.

1. $\widetilde{I}$. holds, i.e. $a_1 = 2b_1$. Then, none of $\widetilde{II}$., $\widetilde{III}$., $\widetilde{V}$., and $\widetilde{VII}$ hold. At most one among $\widetilde{IV}$, $\widetilde{VI}$., $\widetilde{VIII}$., $\widetilde{IX}$, $\widetilde{X}$., $\widetilde{XI}$. holds or $\widetilde{IV}. = \widetilde{XI}$. holds. The latter case is the only problematic case. In this case, deduce from equation (18) and Lemma B.1 that $\nu_{41} = \nu_{42} = 0$. Also, we have that (13) holds. However, by supposing λ_0 known, we have already shown that (13) cannot hold.

2. Suppose $\widetilde{II}.$ holds, i.e. $a_2 = 2b_1$. Then, none of $\tilde{I}.$, $\widetilde{IV}.$, $\widetilde{VI}.$, $\widetilde{VII}.$, $\widetilde{VIII}.$, $\widetilde{IX}.$ hold. At most one among $\widetilde{III}.$, $\widetilde{VI}.$, $\widetilde{X}.$, and $\widetilde{XI}.$ holds. The result follows.

3. Suppose $\widetilde{III}.$ holds, i.e. $a_1 = b_1 + b_2$. Then, none of $\tilde{I}.$, $\widetilde{IV}.$, $\tilde{V}.$, $\widetilde{VII}.$ hold. At most one among $\widetilde{II}.$,$\widetilde{VI}.$,$\widetilde{VIII}.$,$\widetilde{IX}.$,$\widetilde{X}.$,$\widetilde{XI}.$ holds. The result follows.

4. Suppose $\widetilde{IV}.$ holds, i.e. $a_2 = b_1 + b_2$. Then, none of $\widetilde{II}.$,$\widetilde{III}.$,$\widetilde{VI}.$,$\widetilde{VII}.$,$\widetilde{VIII}.$,$\widetilde{IX}.$,$\widetilde{X}.$ hold. At most one among $\tilde{I}.$, $\tilde{V}.$, $\widetilde{XI}.$ holds or $\tilde{I}. = \widetilde{XI}.$ holds. The only problematic case is when $\tilde{I}. = \widetilde{IV}.$ and we are back to subcase 2.

5. Suppose $\tilde{V}.$ holds, i.e. $a_1 = 2b_2 - b_1$. Then, none of $\tilde{I}.$,$\widetilde{III}.$,$\widetilde{VI}.$ and $\widetilde{VII}.$ hold. At most one among $\widetilde{II}.$, $\widetilde{IV}.$, $\widetilde{VIII}.$, $\widetilde{IX}.$, $\widetilde{X}.$, $\widetilde{XI}.$ holds or $\tilde{I}. = \widetilde{IX}.$ holds. The only no trivial case is when $\tilde{I}. = \widetilde{IX}.$ holds. In this case, from equation (18) and Lemma B.1, we have $\nu_{41} = \nu_{42} = 0$. Now, from equation (17) and Lemma B.1, we have $\nu_{31} = \nu_{32}$. From equation (20) and Lemma B.1 we have $\nu_{21} = \nu_{22} = 0$. By Lemma B.1 again, conclude that $\nu_{11} = \nu_{12} = 0$.

6. Suppose $\widetilde{VI}.$ holds, i.e. $a_2 = 2b_2 - b_1$. Then, none of $\widetilde{II}.$, $\widetilde{IV}.$, $\tilde{V}.$, $\widetilde{VII}.$, $\widetilde{VIII}.$ and $\widetilde{XI}.$ hold. At most one among $\tilde{I}.$,$\widetilde{III}.$,$\widetilde{IX}.$,$\widetilde{X}.$ holds. The result follows.

7. Suppose $\widetilde{VII}.$ holds, i.e. $a_1 = 2b_2$. Then, none of $\tilde{I}.$,$\widetilde{II}.$,$\widetilde{III}.$,$\widetilde{IV}.$,$\tilde{V}.$,$\widetilde{VI}.$,$\widetilde{VIII}.$ hold. At most one among $\widetilde{IX}.$, $\widetilde{X}.$ and $\widetilde{XI}.$ holds. The result follows.

8. Suppose $\widetilde{VIII}.$ holds, i.e. $a_2 = 2b_2$. Then, none of $\widetilde{II}.$, $\widetilde{IV}.$, $\widetilde{VI}.$, $\widetilde{VII}.$ and $\widetilde{X}.$ hold. At most one among $\tilde{I}.$, $\widetilde{III}.$, $\widetilde{IV}.$, $\widetilde{IX}.$ and $\widetilde{XI}.$ holds or ($\widetilde{III}.$ and $\widetilde{XI}. = \widetilde{VIII}.$) holds or ($\tilde{V}.$ and $\widetilde{IX}. = \widetilde{VIII}.$) holds. The only problematic case is when $\widetilde{III}.$ and $\widetilde{XI}. = \widetilde{VIII}.$ holds. In this case, deduce from equation (17) and Lemma B.1 that $\nu_{31} = \nu_{32} = 0$. Also,

$$\begin{cases} a_1 &= b_1 + b_2, \\ a_2 &= 2b_2. \end{cases} \tag{21}$$

The unique pairwise different exponents of the exponential polynomial in (1) are: b_2, $2b_2 (= a_2)$, $b_1 + b_2 (= a_1)$, $b_1 + a_1$, $b_2 + a_1 (= b_1 + a_2)$, and $b_2 + a_2$.

This put six restrictions on $\boldsymbol{\nu}\backslash\{\nu_{31},\nu_{32}\}$:

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & \alpha_2 e^{b_2\gamma_2} & \alpha_2 e^{b_2\gamma_1} \\ 1 & 1 & 0 & 0 & \alpha_1 e^{b_1\gamma_2} & \alpha_1 e^{b_1\gamma_1} \\ \alpha_1 e^{b_1\gamma_2} & \alpha_1 e^{b_1\gamma_1} & 0 & 0 & 0 & 0 \\ \alpha_2 e^{b_2\gamma_2} & \alpha_2 e^{b_2\gamma_1} & \alpha_1 e^{b_1\gamma_2} & \alpha_1 e^{b_1\gamma_1} & 0 & 0 \\ 0 & 0 & \alpha_2 e^{b_2\gamma_2} & \alpha_2 e^{b_2\gamma_1} & & \end{pmatrix} \cdot \begin{pmatrix} \nu_{11} \\ \nu_{12} \\ \nu_{21} \\ \nu_{22} \\ \nu_{41} \\ \nu_{42} \end{pmatrix} = \mathbf{0}.$$

A tedious computation shows that the determinant of the above matrix is zero meaning that this system of linear equations has non trivial solutions. However, notice that (21) cannot hold since

$$(21) \iff \begin{cases} \beta_k & = (1+\lambda_{02})\beta_{0k} \\ \beta_k & = 2\beta_{0k} \end{cases} \implies \lambda_{02} = 1,$$

and $(1,1) \notin \Lambda_\tau$.

9. Suppose $\widetilde{IX}$. holds, i.e. $a_2 = b_1 + a_1$. Then, none of $\widetilde{II}$., $\widetilde{IV}$., $\widetilde{X}$. and $\widetilde{XI}$. hold. At most one among $\widetilde{I}$., $\widetilde{III}$., $\widetilde{V}$., $\widetilde{VI}$., $\widetilde{VII}$., $\widetilde{VIII}$., $\widetilde{IX}$. holds or ($\widetilde{V}$. and $\widetilde{IX}. = \widetilde{VIII}$.) holds. The problematic case is when ($\widetilde{V}$. and $\widetilde{IX}. = \widetilde{VIII}$.) holds. Then, we are back to subcase 9.

10. Suppose $\widetilde{X}$. holds, i.e. $a_2 = b_2 + a_1$. Then, none of $\widetilde{IV}$., $\widetilde{VIII}$., $\widetilde{IX}$. and $\widetilde{XI}$. hold. At most one among $\widetilde{I}$., $\widetilde{II}$., $\widetilde{III}$., $\widetilde{V}$., $\widetilde{VI}$. and $\widetilde{VII}$. holds. The result follows.

11. Suppose $\widetilde{XI}$. holds, i.e. $a_2 = b_2 - b_1$. Then, none of $\widetilde{VI}$., $\widetilde{IX}$. and $\widetilde{X}$. hold. At most one among $\widetilde{I}$., $\widetilde{II}$., $\widetilde{III}$., $\widetilde{IV}$., $\widetilde{V}$., $\widetilde{VII}$. and $\widetilde{VIII}$. holds. The result follows.

- Third, suppose $0 < a_1 < b_1 < a_2 < b_2$. Then, we have

$$
\begin{aligned}
b_1 &< b_1 + b_2 \\
&< 2b_2 \\
&< 2b_1 \\
&< b_1 + a_1 \\
&< b_1 + a_2 \\
&< b_2 + a_1 \\
&< b_2 + a_2 \\
&< b_2 \\
&> a_1 \\
&< a_2
\end{aligned}
,\qquad
\begin{aligned}
2b_1 &< b_1 + b_2 \\
&< 2b_2 \\
&> b_1 \\
&> b_1 + a_1 \\
&< b_1 + a_2 \\
&\stackrel{?}{=} b_2 + a_1 \quad (a.) \\
&\stackrel{?}{=} b_2 + a_2 \quad (b.) \\
&\neq b_2 \\
&> a_1 \\
&\stackrel{?}{=} a_2 \quad (c.)
\end{aligned}
,\qquad
\begin{aligned}
b_1 + b_2 &> b_1 \\
&> b_2 \\
&> a_1 \\
&> a_2 \\
&< 2b_2 \\
&> 2b_1 \\
&> b_1 + a_1 \\
&> b_1 + a_2 \\
&> b_2 + a_1 \\
&< b_2 + a_2
\end{aligned}
,
\tag{22}
$$

and

$$
\begin{aligned}
b_2 &< b_1 + b_2 \\
&< 2b_2 \\
&\neq 2b_1 \\
&\stackrel{?}{=} b_1 + a_1 \quad (d.) \\
&\stackrel{?}{=} b_1 + a_2 \quad (e.) \\
&< b_2 + a_1 \\
&< b_2 + a_2 \\
&> b_1 \\
&> a_1 \\
&> a_2
\end{aligned}
,\qquad
\begin{aligned}
2b_2 &> b_1 + b_2 \\
&> 2b_1 \\
&> b_1 \\
&> b_1 + a_1 \\
&> b_1 + a_2 \\
&> b_2 + a_1 \\
&> b_2 + a_2 \\
&> b_2 \\
&> a_1 \\
&> a_2
\end{aligned}
,\qquad
\begin{aligned}
b_1 + b_2 &> b_1 \\
&> b_2 \\
&> a_1 \\
&> a_2 \\
&< 2b_2 \\
&> 2b_1 \\
&> b_1 + a_1 \\
&> b_1 + a_2 \\
&> b_2 + a_1 \\
&< b_2 + a_2
\end{aligned}
,
\tag{23}
$$

and

$$
\begin{aligned}
a_1 &< b_1 \\
&< b_2 \\
&< b_1+b_2 \\
&< 2b_2 \\
&< 2b_1 \\
&< b_1+a_1 \\
&< b_1+a_2 \\
&< b_2+a_1 \\
&< b_2+a_2 \\
&< a_2
\end{aligned}
,\qquad
\begin{aligned}
b_1+a_1 &< b_1+b_2 \\
&< 2b_2 \\
&< 2b_1 \\
&> b_1 \\
&< b_1+a_2 \\
&< b_2+a_1 \\
&< b_2+a_2 \\
&\stackrel{?}{=} b_2 \quad (d.) \\
&> a_1 \\
&\stackrel{?}{=} a_2 \quad (f.).
\end{aligned}
,\qquad
\begin{aligned}
b_2+a_1 &< b_1+b_2 \\
&> b_1 \\
&> b_2 \\
&> a_1 \\
&> a_2 \\
&< 2b_2 \\
&\stackrel{?}{=} 2b_1 \quad (a.) \\
&> b_1+a_1 \\
&\stackrel{?}{=} b_1+a_2 \quad (g.) \\
&< b_2+a_2
\end{aligned}
\tag{24}
$$

and

$$
\begin{aligned}
a_2 &> b_1 \\
&< b_2 \\
&< b_1+b_2 \\
&< 2b_2 \\
&\stackrel{?}{=} 2b_1 \quad (c.) \\
&\stackrel{?}{=} b_1+a_1 \quad (f.) \\
&< b_1+a_2 \\
&< b_2+a_1 \\
&< b_2+a_2 \\
&> a_1
\end{aligned}
,\qquad
\begin{aligned}
b_1+a_2 &< b_1+b_2 \\
&< 2b_2 \\
&> b_1 \\
&> b_1+a_1 \\
&\stackrel{?}{=} b_2+a_1 \quad (g.) \\
&< b_2+a_2 \\
&\stackrel{?}{=} b_2 \quad (e.) \\
&> 2b_1 \\
&> a_1 \\
&> a_2
\end{aligned}
,\qquad
\begin{aligned}
b_2+a_2 &> b_1+b_2 \\
&> b_1 \\
&> b_2 \\
&> a_1 \\
&> a_2 \\
&< 2b_2 \\
&\stackrel{?}{=} 2b_1 \quad (b.) \\
&> b_1+a_1 \\
&> b_1+a_2 \\
&> b_2+a_1
\end{aligned}
.
\tag{25}
$$

There are seven subcases left.

1. a. holds, i.e. $a_1 = 2b_1 - b_2$. Then, b., c., d. cannot hold. Also, at most one equation among e., f., g. holds. The result follows.
2. b. holds, i.e. $a_2 = 2b_1 - b_2$. Then, none of a., c., e., f. and g. hold. The result follows.
3. Suppose c. holds, i.e. $a_2 = 2b_1$. Then, none of a., b., e., f. and g. hold. The result follows.

4. Suppose $d.$ holds, i.e. $a_1 = b_2 - b_1$. Then, none of $a.$, $e.$, $f.$ and $g.$ holds. At most one among $b.$ and $c.$ holds. The result follows.
5. Suppose $e.$ holds, i.e. $a_2 = b_2 - b_1$. Then, none of $b.$, $c.$ and $d.$ hold. At most one among $a.$, $f.$ and $g.$ holds. The result follows.
6. Suppose $f.$ holds, i.e. $a_2 = b_1 + a_1$. Then, none of $b.$, $c.$, $d.$ and $g.$ hold. At most one among $a.$ and $e.$ holds. The result follows.
7. Suppose $g.$ holds, i.e. $a_2 = b_1 - b_2 + a_1$. Then, none of $b.$, $c.$, $d.$ and $f.$ hold. At most one among $a.$ and $e.$ holds. The result follows.

- Fourth, suppose $0 < a_1 < a_2 < b_1 < b_2$. Then, we have

$$\begin{aligned} b_1 &< b_1 + b_2 \\ &< 2b_2 \\ &< 2b_1 \\ &< b_1 + a_1 \\ &< b_1 + a_2 \\ &< b_2 + a_1 \\ &< b_2 + a_2 \\ &< b_2 \\ &> a_1 \\ &> a_2 \end{aligned}, \qquad \begin{aligned} 2b_1 &< b_1 + b_2 \\ &< 2b_2 \\ &> b_1 \\ &> b_1 + a_1 \\ &> b_1 + a_2 \\ &\stackrel{?}{=} b_2 + a_1 \quad (one) \\ &\stackrel{?}{=} b_2 + a_2 \quad (two) \\ &\neq b_2 \\ &> a_1 \\ &> a_2 \end{aligned}, \qquad \begin{aligned} b_1 + b_2 &> b_1 \\ &> b_2 \\ &> a_1 \\ &> a_2 \\ &< 2b_2 \\ &> 2b_1 \\ &> b_1 + a_1 \\ &> b_1 + a_2 \\ &> b_2 + a_1 \\ &> b_2 + a_2 \end{aligned}, \tag{26}$$

and

$$\begin{aligned} b_2 &< b_1 + b_2 \\ &< 2b_2 \\ &\neq 2b_1 \\ &\stackrel{?}{=} b_1 + a_1 \quad (three) \\ &\stackrel{?}{=} b_1 + a_2 \quad (four) \\ &< b_2 + a_1 \\ &< b_2 + a_2 \\ &> b_1 \\ &> a_1 \\ &> a_2 \end{aligned}, \qquad \begin{aligned} 2b_2 &> b_1 + b_2 \\ &> 2b_1 \\ &> b_1 \\ &> b_1 + a_1 \\ &> b_1 + a_2 \\ &\neq b_2 + a_1 \\ &\neq b_2 + a_2 \\ &\neq b_2 \\ &> a_1 \\ &> a_2 \end{aligned}, \qquad \begin{aligned} b_1 + b_2 &> b_1 \\ &> b_2 \\ &> a_1 \\ &> a_2 \\ &< 2b_2 \\ &> 2b_1 \\ &> b_1 + a_1 \\ &> b_1 + a_2 \\ &> b_2 + a_1 \\ &> b_2 + a_2 \end{aligned}, \tag{27}$$

and

$$
\begin{array}{ll}
a_1 & < b_1 \\
 & < b_2 \\
 & < b_1 + b_2 \\
 & < 2b_2 \\
 & < 2b_1 \\
 & < b_1 + a_1 \\
 & < b_1 + a_2 \\
 & < b_2 + a_1 \\
 & < b_2 + a_2 \\
 & < a_2
\end{array},
\quad
\begin{array}{ll}
b_1 + a_1 & < b_1 + b_2 \\
 & < 2b_2 \\
 & < 2b_1 \\
 & > b_1 \\
 & < b_1 + a_2 \\
 & < b_2 + a_1 \\
 & < b_2 + a_2 \\
 & \stackrel{?}{=} b_2 \quad (three) \\
 & > a_1 \\
 & \stackrel{?}{=} a_2 \quad (five)
\end{array},
\quad
\begin{array}{ll}
b_2 + a_1 & < b_1 + b_2 \\
 & > b_1 \\
 & > b_2 \\
 & > a_1 \\
 & > a_2 \\
 & < 2b_2 \\
 & \stackrel{?}{=} 2b_1 \quad (one) \\
 & > b_1 + a_1 \\
 & \stackrel{?}{=} b_1 + a_2 \quad (six) \\
 & < b_2 + a_2
\end{array},
\tag{28}
$$

and

$$
\begin{array}{ll}
a_2 & < b_1 \\
 & < b_2 \\
 & < b_1 + b_2 \\
 & < 2b_2 \\
 & < 2b_1 \\
 & \stackrel{?}{=} b_1 + a_1 \quad (five) \\
 & < b_1 + a_2 \\
 & < b_2 + a_1 \\
 & < b_2 + a_2 \\
 & > a_1
\end{array},
\quad
\begin{array}{ll}
b_1 + a_2 & > b_1 + b_2 \\
 & < 2b_2 \\
 & > b_1 \\
 & > b_1 + a_1 \\
 & \stackrel{?}{=} b_2 + a_1 \quad (six) \\
 & < b_2 + a_2 \\
 & \stackrel{?}{=} b_2 \quad (four) \\
 & > 2b_1 \\
 & > a_1 \\
 & > a_2
\end{array},
\quad
\begin{array}{ll}
b_2 + a_2 & > b_1 + b_2 \\
 & > b_1 \\
 & > b_2 \\
 & > a_1 \\
 & > a_2 \\
 & < 2b_2 \\
 & \stackrel{?}{=} 2b_1 \quad (two) \\
 & > b_1 + a_1 \\
 & > b_1 + a_2 \\
 & > b_2 + a_1
\end{array}.
\tag{29}
$$

There are six subcases left.

1. Suppose *one*. holds, i.e. $a_1 = 2b_1 - b_2$. Then, *two*. or *three*. cannot hold and at most one equation out of *four*., *five*. and *six*. holds. The result follows.

2. Suppose *two*. holds, i.e. $a_2 = 2b_1 - b_2$. Then, *one*., *four*. and *five*. cannot hold and at most one equation out of *three*. and *six*. holds. The result follows.

3. Suppose *three*. holds, i.e. $a_1 = b_2 - b_1$. Then, none of *one*., *four*. and *five*. hold. At most one among *two*. and *six*. hold. The result follows.

4. Suppose *four*. holds, i.e. $a_2 = b_2 - b_1$. Then, none of *two*., *three*. and *six*. hold. At most one among *five*. and *one*. hold and the result follows.

5. Suppose *five*. holds, i.e. $a_2 = b_1 + a_1$. Then, none of *two*., *three*. and *six*. hold. At most one among *one*. and *four*. holds. The result follows.

6. Suppose *six*. holds, i.e. $a_2 = b_2 - b_1 + a_1$. Then, none of *one*., *four*. and *five*. hold. At most one among *two*. and *three*. holds. The result follows.

Case 3 : $\beta_{0k} < 0$ **and** $\beta_k > 0$**.** Then, we necessarily have $b_2 < b_1 < 0 < a_1 < a_2$ and the following order relations hold.

$$\begin{aligned} b_1 &> b_1 + b_2 \\ &> 2b_2 \\ &> 2b_1 \\ &< b_1 + a_1 \\ &< b_1 + a_2 \\ &\stackrel{?}{=} b_2 + a_1 \quad (i.) \\ &\stackrel{?}{=} b_2 + a_2 \quad (ii.) \\ &> b_2 \\ &< a_1 \\ &< a_2. \end{aligned} \quad , \quad \begin{aligned} 2b_1 &> b_1 + b_2 \\ &> 2b_2 \\ &< b_1 \\ &< b_1 + a_1 \\ &< b_1 + a_2 \\ &\stackrel{?}{=} b_2 + a_1 \quad (iii.) \\ &\stackrel{?}{=} b_2 + a_2 \quad (iv.) \\ &\neq b_2 \\ &< a_1 \\ &< a_2 \end{aligned} \quad , \quad \begin{aligned} b_1 + b_2 &< b_1 \\ &< b_2 \\ &< a_1 \\ &< a_2 \\ &> 2b_2 \\ &< 2b_1 \\ &< b_1 + a_1 \\ &< b_1 + a_2 \\ &< b_2 + a_1 \\ &< b_2 + a_2 \end{aligned} \tag{30}$$

and

$$
\begin{aligned}
b_2 &> b_1 + b_2 \\
&> 2b_2 \\
&\neq 2b_1 \\
&< b_1 + a_1 \\
&< b_1 + a_2 \\
&< b_2 + a_1 \\
&< b_2 + a_2 \\
&< b_1 \\
&< a_1 \\
&< a_2
\end{aligned}
,\qquad
\begin{aligned}
2b_2 &< b_1 + b_2 \\
&< 2b_1 \\
&< b_1 \\
&< b_1 + a_1 \\
&< b_1 + a_2 \\
&< b_2 + a_1 \\
&< b_2 + a_2 \\
&< b_2 \\
&< a_1 \\
&< a_2
\end{aligned}
,\qquad
\begin{aligned}
b_1 + b_2 &< b_1 \\
&< b_2 \\
&< a_1 \\
&< a_2 \\
&> 2b_2 \\
&< 2b_1 \\
&< b_1 + a_1 \\
&< b_1 + a_2 \\
&< b_2 + a_1 \\
&< b_2 + a_2
\end{aligned}
, \tag{31}
$$

and

$$
\begin{aligned}
a_1 &> b_1 \\
&> b_2 \\
&> b_1 + b_2 \\
&> 2b_2 \\
&> 2ba_1 \\
&> b_1 + a_1 \\
&\stackrel{?}{=} b_1 + a_2 \quad (v.) \\
&> b_2 + a_1 \\
&\stackrel{?}{=} b_2 + a_2 \quad (vi.) \\
&< a_2
\end{aligned}
,\qquad
\begin{aligned}
b_1 + a_1 &> b_1 + b_2 \\
&> 2b_2 \\
&> 2b_1 \\
&> b_1 \\
&< b_1 + a_2 \\
&> b_2 + a_1 \\
&\stackrel{?}{=} b_2 + a_2 \quad (vii.) \\
&> b_2 \\
&< a_1 \\
&< a_2
\end{aligned}
,\qquad
\begin{aligned}
b_2 + a_1 &> b_1 + b_2 \\
&\stackrel{?}{=} b_1 \quad (i.) \\
&> b_2 \\
&< a_1 \\
&< a_2 \\
&> 2b_2 \\
&\stackrel{?}{=} 2b_1 \quad (iii.) \\
&< b_1 + a_1 \\
&< b_1 + a_2 \\
&< b_2 + a_2
\end{aligned}
, \tag{32}
$$

and

$$
\begin{aligned}
a_2 &> b_1 \\
&> b_2 \\
&> b_1 + ba_2 \\
&> 2b_2 \\
&> 2b_1 \\
&> b_1 + a_1 \\
&> b_1 + a_2 \\
&> b_2 + a_1 \\
&> b_2 + a_2 \\
&> a_1
\end{aligned}
,\qquad
\begin{aligned}
b_1 + a_2 &> b_1 + b_2 \\
&> 2b_2 \\
&> b_1 \\
&> b_1 + a_1 \\
&> b_2 + a_1 \\
&> b_2 + a_2 \\
&> b_2 \\
&> 2b_1 \\
&\overset{?}{=} a_1 \quad (v.) \\
&> a_2.
\end{aligned}
,\qquad
\begin{aligned}
b_2 + a_2 &> b_1 + b_2 \\
&\overset{?}{=} b_1 \quad (ii.) \\
&> b_2 \\
&\overset{?}{=} a_1 \quad (vi.) \\
&< a_2 \\
&> 2b_2 \\
&\overset{?}{=} 2b_1 \quad (iv.) \\
&\overset{?}{=} b_1 + a_1 \quad (vii.) \\
&< b_1 + a_2 \\
&> b_2 + a_1
\end{aligned}
. \tag{33}
$$

The potential cases of equality are exactly the same than that in Case 1 and all other inequalities are reversed compared to Case 1. By multiplying each inequality by -1 and reasoning with $-a_1, -a_2, -b_1$ and $-b_2$, we are back to Case 1.

Case 4 : $\beta_{0k} < 0$ and $\beta_k < 0$. By considering $-a_1, -a_2, -b_1$ and $-b_2$, we are back to Case 2.

The proof is completed.